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EUCLIDEAN RECONSTRUCTION IN QUANTUM FIELD THEORY: BETWEEN TEMPERED DISTRIBUTIONS AND FOURIER HYPERFUNCTIONS

by Andreas U. Schmidt

Abstract. In this short note on my talk I want to point out the mathematical difficulties that arise in the study of the relation of Wightman and Euclidean quantum field theory, i.e., the relation between the hierarchies of Wightman and Schwinger functions. The two extreme cases where the reconstructed Wightman functions are either tempered distributions — the well-known Osterwalder-Schrader reconstruction — or modified Fourier hyperfunctions are discussed in some detail. Finally, some perpectives towards a classification of Euclidean reconstruction theorems are outlined and preliminary steps in that direction are presented.

0. Introduction: Why Euclidean Reconstruction

Euclidean methods are widely used in quantum field theory and other areas of mathematical physics. So let me outline some of the reasons for the attractivity of those techniques. I will use the language of axiomatic "Wightman" quantum field theory, for the fundamental notions of which I may refer the reader to my last years talk [1] or the famous books [SW] of Streater and Wightman or [GJ] of Glimm and Jaffe respectively.

To use "Euclidean" methods in quantum field theory amounts formally in changing the coordinates $x = (x^0, x^1, x^2, x^3) = (t, \vec{x})$ of Minkowski space to $\xi = \imath^t x = (it, \vec{x})$, where $\imath = (i, 1, 1, 1)$. By that the original Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$ prescribed by relativistic covariance of the theory becomes the trivial metric of Euclidean space, but the new time coordinate is purely imaginary. The benefits of this transformation is somewhat naïvely but strikingly explained in terms of the *path integral.* This is a formal expression for the so called *generating functional* of a quantum field theory which reads

$$\mathcal{Z}\{f\} = \int \mathcal{D}\Phi \, e^{i\Phi(f) + i\mathcal{L}(f)}.$$

From this expression all *correlation functions* of the theory, i.e., actual probabilities of physical events, can in principle be derived by a formal *variational derivative* of $\mathcal{Z}\{f\}$ with respect to the test function $f \in \mathcal{S}(\mathcal{M})$, which is usually assumed to be tempered, so that the *field* $\Phi \in \mathcal{S}(\mathcal{M})'$ will be a tempered distribution. The exponent in this expression, termed *action integral*, is divided into the *source* part $\Phi(f)$ and the *Lagrangean* $\mathcal{L}(f)$ which is a function (usually a polynomial) of the field Φ and its derivatives. Following the principles of variational theory we know that the *classical* field configurations, i.e., solutions of the Euler-Lagrange equations derived from $\mathcal{L}$, minimize the action, so that their contribution to $\mathcal{Z}\{f\}$ is maximal, whereas fluctuations around the classical situation are supposed to give smaller contributions. This is what essentially allows one to approximate $\mathcal{Z}\{f\}$ by perturbation expansions, e.g., in Planck's constant $\hbar$ (semiclassical approximation) or, most commonly, in a *coupling constant* g appearing in $\mathcal{L}$, which leads to the vast field of *diagrammatic perturbation theory*, see, e.g., [tH]. The free, classical fields around which this expansion takes place are solutions of simple partial differential equations like the wave, Klein-Gordon, and free Dirac equation, in that order:

$$\Box\phi = 0, \quad (\Box + m^2)\phi = 0, \quad (i\gamma^\mu\partial_\mu - m)\phi = 0$$

(the classical field ϕ should not be mixed up with the operator valued distribution Φ). These equations follow as Euler-Lagrange equations from the *free* part $\mathcal{L}_\mathrm{f}$ of the Lagrangean $\mathcal{L} = \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{i}$.

The mathematical properties of $\mathcal{Z}\{f\}$ are not very satisfactory: The *path space measure* $\mathcal{D}\Phi$ over the space of all possible field configurations cannot be defined because of the unknown geometrical structure of this space, and even if it would be well-defined, the integral would by no means be guaranteed to converge. The situation changes dramatically after the seemingly innocent change to Euclidean coordinates. The new generating functional

$$\mathfrak{S}\{f\} = \int d\mu \, e^{-\Phi(f) - \mathcal{L}(f)}$$

is an integral with respect to a well-defined *Borel probability measure* $d\mu$ on $\mathcal{S}(\mathbf{R}^4)'$ and converges due to the negative definite damping factor in the exponent (usually like a Gaussian).

On the level of correlation functions, which are derived from the generating functionals or are the input of the axiomatic approach, the advantage of working in the Euclidean regime is reflected by the fact that the hierarchy of Wightman n-point functions $\mathfrak{W}_n$, $\mathbf{W}_n$, are tempered distributions, whereas their Euclidean equivalents — the so-called *Schwinger functions* — are indeed *real analytic functions.*

So from a general point of view, we have turned a quantum theory into a theory which is nothing but a formulation of *statistical mechanics* by a simple change of coordinates. It is not surprising that very commonly physicists use the Euclidean framework to do actual calculations, transferring their results back to the quantum regime by writing t for it again. However, somewhere in this "backtransformation" the nontrivial quantum nature should reappear, which means in terms of correlation functions that the Schwinger functions should gain singularities in the process of analytic continuation back to Minkowski spacetime. So, besides the mathematical justification of the use of Euclidean methods, a consistent theory of what is called *Euclidean reconstruction* of the Wightman hierarchy from the Schwinger hierarchy would help to answer fundamental questions about quantum field theory in the axiomatic framework. Let us first walk the simpler path, namely the continuation of Wightman functions to the Euclidean domain.

1. Analytic continuation of Wightman Functions to the Euclidean Domain

Starting from the Wightman functions (in difference variables) $\mathbf{W}_{n+1} \in \mathcal{S}(\mathbf{R}^{4n})$ which are distributions in the real variables $x_1, \ldots, x_n$, $x_i = (x_i^0, \vec{x}_i) \in \mathbf{R}^4$, we want to see that they can be extended to holomorphic functions on some domain in $\mathbf{C}^{4n}$ and determine the shape of that domain. In this process, the physical axioms imposed on the Wightman functions (see [1]) enter at different stages.

In the first step we use the fact that the Fourier transforms $\widetilde{\mathbf{W}}_{n+1}$ have their support in the n-fold product of the forward lightcone by the *spectrum condition*: $\mathrm{supp}\widetilde{\mathbf{W}}_{n+1} \subset \overline{V}^{+n}$, $\overline{V}^{+} \equiv \{x \in \mathcal{M} \mid (x,x) \geq 0, x^0 \geq 0\}$, where $(x,x) = {}^t x \eta x$ denotes the usual Minkowski inner product. By that fact and a general theorem on the Laplace transformation of tempered distributions, see, e.g., [RS], theorem IX.16, $\mathbf{W}_{n+1}$ are boundary values in the sense of $\mathcal{S}(\mathbf{R}^{4n})'$ of holomorphic functions in the *forward tube* $T_n \equiv \mathbf{R}^{4n} + iV^{+n}$. This theorem is a generalization of the famous Payley-Wiener-Schwartz theorem for distributions with compact support to the tempered case. The characterizing property of the Fourier-Laplace transforms of tempered distributions with support in some cone is that they are polynomially bounded. This will be of crucial importance in the Euclidean reconstruction of tempered distributions, as we will see below.

Next one uses the *invariance* of the Wightman functions under the action of the *Lorentz group*. This is the connected component $\mathrm{L}(\mathbf{R})$ containing the identity of

$$\{g \in \mathrm{GL}(4,\mathbf{R}) \mid (gx,gx) = (x,x) \text{ for } x \in \mathbf{R}^4\},$$

where the action of $\mathrm{L}(\mathbf{R})$ is extended to $\mathbf{R}^{4n}$ in the obvious way. The key is the following

THEOREM [BARGMANN-HALL-WIGHTMAN]. *If a function $\mathbf{W}_{n+1}$ is holomorphic on the tubular domain T_n and invariant under the action of the Lorentz group then $\mathbf{W}_{n+1}$ can be analytically continued to a single valued function on the* extended tube

$$T_n' \equiv \bigcup_{g\in\mathrm{L}(\mathbf{C})} gT_n \subset \mathbf{C}^{4n},$$

where $\mathrm{L}(\mathbf{C})$ *denotes the* complexification *of the Lorentz group, and $\mathbf{W}_{n+1}$ is invariant under the action of* $\mathrm{L}(\mathbf{C})$.

This may be found, e.g., in [SW], theorem 2.11. The difficult point in its proof is the single-valuedness of the analytic continuation.

Unlike the tubular domain T_n, the extended tube T_n' contains real points. These points have the remarkable property of being alltogether *spacelike*: We call $x \in \mathbf{R}^4$ *spacelike* if $(x,x) < 0$, *timelike* if $(x,x) > 0$ and finally $x = (x_1,\dots,x_n) \in \mathbf{R}^{4n}$ is called a *Jost point* if it has the property that $\lambda_1 x_1 + \dots + \lambda_n x_n$ is spacelike for any $\lambda_j \geq 0$ with $\sum_{j=1}^n \lambda_j > 0$. A fundamental lemma due to Jost is

LEMMA [JOST]. *The set $T_n' \cap \mathbf{R}^{4n}$ is open and we have*

$$T_n' \cap \mathbf{R}^{4n} = \textit{ the totality of Jost points of } \mathbf{R}^{4n}.$$

An accessible proof may be found in [MO], chapter 9, §6. Combining this with the above theorem we find that the $\mathbf{W}_{n+1}$ are real analytic functions on the set of Jost points.

Now remember that the $\mathbf{W}_{n+1}$ in the difference variables are defined by the Wightman functions $\mathfrak{W}_{n+1}$ in the ordinary variables, for which we simply use the letter y, by

$$\mathfrak{W}_{n+1}(y_1,\dots,y_{n+1}) = \mathbf{W}_{n+1}(y_2 - y_1,\dots,y_{n+1} - y_n),$$

and that *causality* is imposed on the $\mathfrak{W}_{n+1}$:

$$\mathfrak{W}(y_1, \dots, y_j, y_{j+1}, \dots, y_{n+1}) = \mathfrak{W}(y_1, \dots, y_{j+1}, y_j, \dots, y_{n+1}),$$

whenever $x_j = y_j - y_{j+1}$ is spacelike. Now in the Jost points all differences are spacelike so that we can apply any permutation to the $n+1$ variables $y_1, \dots, y_{n+1}$ and will obtain the same analytic function $\mathbf{W}_{n+1}$ when going back to difference variables. So, finally the $\mathbf{W}_{n+1}$ can be uniquely extended to the *permuted extended tube* defined by

$$T_n'^{\,\Pi} \equiv \{\pi x \mid x \in T_n',\ \pi \in S_{n+1}\},$$

where for $x = (y_2 - y_1, \dots, y_n - y_{n-1})$ and a permutation $\pi \in S_{n+1}$ we define $\pi x \equiv (y_{\pi(2)} - y_{\pi(1)}, \dots, y_{\pi(n+1)} - y_{\pi(n)})$.

The last step will be to show that $T_n'^{\,\Pi}$ contains the Euclidean points. Let us first state what we mean by these: A point $x = (y_2 - y_1, \dots, y_{n+1} - y_n) \in \mathbf{C}^{4n}$ is called *Euclidean point* if it is of the form $(\imath y_2' - \imath y_1', \dots, \imath y_{n+1}' - \imath y_n')$, with $(y_1', \dots, y_{n+1}') \in \mathbf{R}^{4(n+1)}$, i.e., it has purely imaginary time and purely real space components. An Euclidean point is called *non-coincident* if $y_j \neq y_k$ for all $j \neq k$.

PROPOSITION. *The permuted extended tube $T_n'^{\,\Pi}$ contains all the non-coincident Euclidean points of $\mathbf{C}^{4n}$.*

We sketch the proof following [BO], proposition 9.10:

PROOF. Begin with the simplest situation, where all the time components $y_j'^0$ are distinct for $1 \leq j \leq n+1$. Then, by a suitable permutation π which arranges the y_j' such that $y_{\pi(j)}'^0$ increases when j increases, we find that the difference vector x determined by $\imath \pi y'$ lies in T_n since $y_{\pi(j)}'^0 - y_{\pi(j+1)}'^0 > 0$, which means that $\imath$ applied to all these difference vectors lies in the forward tube. In the general situation we may nevertheless assume without loss of generality $y_1'^0 \leq \dots \leq y_{n+1}'^0$, by applying a suitable permutation if necessary. Consider the collection of vectors in $\mathbf{R}^3$

$$\{e_{jk} \equiv y_j' - y_k' \mid y_j'^0 = y_k'^0,\ j,k = 1, \dots, n+1,\ j \neq k\}.$$

Since all these vectors are non-zero on non-coincident points we may find a three-dimensional unit vector $\vec{s}$ that is not orthogonal to any of them. Assume, again without loss of generality, that $\vec{s}$ is a vector along the y'^3-axis. Then we may well-order the set $\{y_j'\}$ according to their y'^3-component. Now the Lorentz-transformation

$$\Lambda_\beta = \begin{pmatrix} \cos\beta & 0 & 0 & \sin\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin\beta & 0 & 0 & \cos\beta \end{pmatrix}$$

rotates the difference vector x corresponding to $\imath y'$ into $T_n \subset {T'_n}^{\Pi}$ for suitable β.

We call the restriction of the Wightman functions to the non-coincident Euclidean region the *Schwinger functions* of Euclidean quantum field theory and denote them by $\mathfrak{S}_n$ in the ordinary and $\mathbf{S}_n$ in the difference variables. As we have shown above, the $\mathfrak{S}_n$ are symmetric in their arguments, so that it is enough to consider the $\mathbf{S}_n$ in the difference variables as functions on $\imath\mathbf{R}_+^{4(n-1)} \equiv (i\mathbf{R}_+ \times \mathbf{R}^3)^{n-1}$:

$$\mathbf{S}_n \equiv \mathbf{W}_n|_{\imath\mathbf{R}_+^{4(n-1)}}, \quad \mathfrak{S}_n \equiv \mathfrak{W}_n(\imath y_1, \dots, \imath y_n).$$

Let us finally list the properties of the Schwinger functions:

- Real analyticity,
- Euclidean, i.e., SO_4-covariance,
- positivity, an equivalent of axiom B in [1], and
- total symmetry.

Further axioms on the relativistic side, like the *cluster decomposition property* (axiom C in [1]), will in general also be reflected by counterparts on the Euclidean side, but see [SI] for a deeper discussion.

2. Reconstruction of Tempered Distributions

The reconstruction of Wightman functions as tempered distributions was done by Osterwalder and Schrader in [2]. Although the basic idea — to use the Euclidean covariance of the Schwinger functions to continue them analytically from their original domain of definition $\mathbf{C}_+^{4n}$ — is quite simple, the actual reconstruction of tempered distributions turned out to be a great effort. The point is the following:

In the final step of reconstruction a Payley-Wiener-Schwartz-like theorem has to be used to yield the Fourier transforms of the Wightman functions as boundary values in the sense of $\mathcal{S}'$ of the Fourier-Laplace transforms of the analytically continued Schwinger functions. As we have already stated above, for such a theorem to be applicable, the Schwinger functions must satisfiy certain polynomial growth conditions. The difficulty lies in controling these growth conditions in the process of analytic continuation. Let us sketch this somewhat formally:

The Schwinger function $\mathbf{S}_n$ is a real analytic function on $\imath\mathbf{R}_+^{4(n-1)}$ and satisfies a *real estimate* of the form

$$|\mathbf{S}_n(\xi)| \leq A_n^{(0)} \cdot E^{(0)}\left(\xi_j^0, |\vec{\xi_j}|\right), \quad \imath\xi \in \imath\mathbf{R}_+^{4(n-1)}. \tag{RE}$$

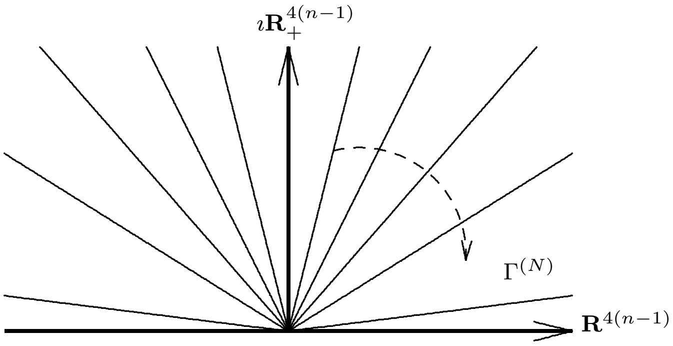
Indeed such an estimate is the basis for the *equivalence* of relativistic and Euclidean theory: The continuations of the Wightman functions naturally satisfy the polynomial bound following from the Payley-Wiener-Schwartz theorem. That is, for the reconstruction of *tempered distributions*, $E^{(0)}$ has the special form

$$E^{(0)} = \left[\left(1 + \max_{1 \leq j \leq n-1} |\vec{\xi}_j| \right) \left(1 + \sum_{j=1}^{n-1} \xi_j^0 \right) \left(1 + \sum_{j=1}^{n-1} (\xi_j^0)^{-1} \right) \right]^{nt} ,$$

for some $t > 0$. The analytic continuation of the time variables back to the real axis can unfortunately not proceed in all $n-1$ variables simultaneously. In fact, this was erroneously assumed in the first version of Osterwalder-Schrader's proof [3], which led to the refinement in [2]. The way they circumvented this difficulty was purely geometrical: In an inductive process they continued $\mathbf{S}_n$ to *cones* $\Gamma^{(N)}$ around the imaginary time axis after N steps. They showed that this cones have increasing opening angle $\pi/2(1 - 2^{-N/2})$ in every coordinate. They then showed that these cones exhaust the whole "upper half plane"

$$\bigcup_{N \in \mathbf{N}} \Gamma^{(N)} = \mathbf{C}_+^{4(n-1)} = \left(\{ \zeta \in \mathbf{C} \mid \operatorname{Im} \zeta > 0 \} \times \mathbf{R}^3 \right)^{(n-1)} ,$$

as shown in the following sketch:



The continued Schwinger functions will then satisfy an estimate very much like (RE) on $\Gamma^{(N)}$:

$$|S_n(\zeta)| \leq A_n^{(N)} \cdot E^{(N)} \left(|\operatorname{Re} \zeta_j^0|, |\operatorname{Im} \zeta_j^0|, |\vec{\zeta}_j| \right), \quad \zeta \in \Gamma^{(N)}, \tag{NE}$$

where $E^{(N)}$ does not differ essentially from $E^{(0)}$, i.e., is also of polynomial form. The difficulty is now that the growth of the constant $A_n^{(N)}$ has to be controled in order that an estimate like (NE) can hold on whole $\mathbf{C}_+^{4(n-1)}$, to make an inverse Payley-Wiener-Schwartz theorem applicable. The solution of Osterwalder-Schrader is to impose an additional condition on the original Schwinger functions, the so-called *linear growth condition*:

$$A_n^{(0)} \leq \alpha (n!)^\beta, \tag{LG}$$

for some constants α, $\beta > 0$[1]. From the linear growth condition, an estimate of $A_n^{(N)}$ follows in the N-th step:

$$A_n^{(N)} \leq n^{\beta n} \cdot 2^{\beta n N},$$

where the factor $n^{\beta n}$ stems directly from (LG). It is a great technical achievement of Osterwalder-Schrader to eliminate N from this estimate by purely geometrical methods! The final form of the estimate is then

$$A_n^{(N)} \leq C_n \leq a b^{n^2},$$

for some a, $b > 0$. This estimate exhibits extremly rapid growth in n, but this is immaterial for the final step of reconstruction, because one is now in the position to apply an theorem of Vladimirov, see [V], pp. 235, which renders the boundary values of the continued Schwinger functions on $\mathbf{R}^{4(n-1)}$ as tempered distributions.

The drawbacks of introducing (LG) are obvious:

- It is merely a sufficient condition for reconstruction of Wightman functions, so that one cannot claim to have full equivalence between the Euclidean and relativistic frameworks. This is also reflected by the fact that
- the reconstructed Wightman functions fulfill additional growth conditions of the form

  $$|\mathfrak{W}_n(f)| \leq \gamma \delta^{n^2} \cdot ||f||,$$

  for some γ, $\delta > 0$ and a certain norm on $\mathcal{S}(\mathbf{R}^{4(n-1)})$.
- The linear growth condition was introduced *ad hoc*.

It was conjectured already by Osterwalder-Schrader that one could overcome these problems by leaving the tempered case and considering Euclidean reconstruction for larger distribution classes. This is what will concern us now.

[1]Notice that in the axiomatic approach to reconstruction the Schwinger functions are not *a priori* assumed to be real analytic functions but also only tempered distributions. So (LG) is formulated in [2] as a modification of the usual temperedness axiom, not directly as a condition for the real analyticity constant $A_n^{(0)}$.

3. Reconstruction of Fourier Hyperfunctions

The mathematical problems of quantum field theory manifest themselves partially in the singularity structure of the correlation functions, see, e.g., [1] and [ST], if one tries to include interaction into the formalism. It was early pointed out by Arthur S. Wightman, see [4] and [5], that this calls for a generalization of distribution theory and several suggestions in this direction have been made, see again [5] and the concise review in [6] and references therein.

One of the most successful choices was to use *hyperfunctions*, which were invented by Mikio Sato in the late 50's and whose fundaments were published in [7] and [8], in a great effort to give rigorous meaning to the notion of "boundary values of holomorphic functions", cf. the introductory texts [KA] and [MO]. Later, also going back to ideas of Sato, the theory was extended by Kawai in his master's thesis and [9] to include Fourier transformation, termed *Fourier hyperfunctions*, and further to (Fourier) hyperfunctions with values in a Hilbert resp. Fréchet space by Yoshifumi Ito and Shigeaki Nagamachi, see [10], [11], and [12] and references therein.

The formulation of axiomatic quantum field theory in terms of Fourier hyperfunctions was carried out by Nagamachi and Nogumichi Mugibayashi in their fundamental papers [13], [14], and [15] and by Erwin Brüning and Nagamachi in [5], see [16], [17], and [18] for concrete physical applications. We will here be mainly outline the results of [15], where an Euclidean reconstruction theorem for the so called *Fourier hyperfunctions of type II*, which are now commonly termed *modified Fourier hyperfunctions* is proven. We can give here only a very coarse sketch of their theory, so the reader may be referred to [19] and [20] for a thorough treatment.

There are generally two ways to define any kind of hyperfunction: The *duality method* in the sense of Schwartz' distribution theory, emphasizing the rôle of the test function spaces, and the *algebro-analytic method*, which views hyperfunctions as boundary values of holomorphic functions. But before going into details, let us fix the topology of the spaces on which our hyperfunctions will live:

Radial Compactification of $\mathbf{C}^n$. To ensure that the Fourier tranfsormation will act as an isomorphism on the test function spaces to be defined below, and by that on the Fourier hyperfunctions, we have to impose what may be loosely speaking called "conditions at infinity." This is only consistently possible if we *compactify* the coordinate space in a certain manner: Let $\mathbf{D}^n \equiv \mathbf{R}^n \sqcup S^{n-1}_\infty$ be the disjoint union of $\mathbf{R}^n$ with the points "at infinity" in every direction. The *complexification* $\mathbf{Q}^n$ is identified with $\mathbf{D}^{2n}$ and is equipped with the following topology: For $x \in \mathbf{C}^n$ take the usual fundamental system of open neighbourhoods. For $x\infty \in S_\infty^{2n-1}$ take open neighbourhoods $x\infty \in C_\infty \subset S_\infty^{2n-1}$ and form their disjoint union with all translations of the cone with vertex at the origin $C \subset \mathbf{C}^n$ that has opening in the direction of C_∞, i.e., $C \cap S^{2n-1} = C_\infty$ under the obvious identification of S_∞^{2n-1} with S^{2n-1}. As a special form for these neighbourhoods we may choose

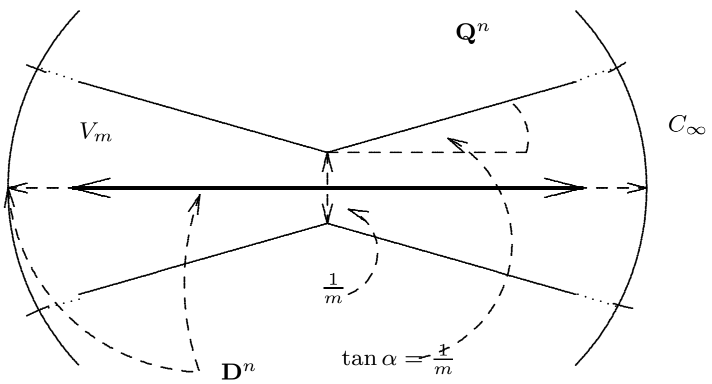
$$C_\delta \sqcup C_\delta\infty, \quad C_\delta \equiv \{z \mid |\operatorname{Im} z| < \delta(|\operatorname{Re} z| + 1)\}.$$

where $C_\delta\infty$ denotes the points "at infinity" of C_δ. We will immediately use these open sets to define test function spaces.

Modified Fourier hyperfunctions as distributions. Choose special neighbourhoods of the real axis $\mathbf{D}^n$ in $\mathbf{Q}^n$ defined by

$$V_m \equiv U_m^n \sqcup (U_m^n)_\infty, \quad U_m \equiv \{z \in \mathbf{C} \mid |\operatorname{Im} z| < (1 + |\operatorname{Re} z|)/m\}, \qquad (V_m)$$

as shown below:



Consider the spaces of *exponentially decreasing holomorphic functions* on these sets defined by

$$\begin{aligned} \mathcal{O}_c^m(V_m) &\equiv \{f \in \mathcal{O}(U_m^n) \mid ||f||_m < \infty\}, \text{ where} \\ ||f||_m &\equiv \sup_{z \in \mathbf{Q}^n \cap V_m} |f(z)| e^{|z|/m}. \end{aligned} \qquad (\mathcal{O}_c^m)$$

Taking the inductive limit of these spaces in the locally convex category with respect to m, i.e., letting the neighbourhoods approach the real axis both in distance and direction, we obtain the space of *rapidly decreasing holomorphic functions* on $\mathbf{D}^n$:

$$\mathcal{P}_{**} \equiv \varinjlim_{m \to \infty} \mathcal{O}_c^m(V_m), \qquad (\mathcal{P}_{**})$$

which is the space of test functions for the modified Fourier hyperfunctions, which we can now define as their dual space $\mathcal{R}(\mathbf{D}^n) \equiv \mathcal{P}'_{**}$.

Modified Fourier hyperfunctions as boundary values. Closer to the original ideas of Sato lies the definition of $\mathcal{R}(\mathbf{D}^n)$ via *relative cohomology* — which comprises the n-dimensional generalization of the notion of boundary values of holomorphic functions. Consider the set of *slowly increasing holomorphic functions* on an open subset $U \subset \mathbf{Q}^n$ given by

$$\widetilde{\widetilde{\mathcal{O}}}(U) \equiv \left\{ f \in \mathcal{O}(U) \,\Big|\, \sup_{z \in \mathbf{Q}^n \cap K} |f(z)| e^{-\varepsilon |z|} < \infty \ \forall \varepsilon > 0,\ K \Subset U \right\}. \tag{SI}$$

Then we can define a presheaf $\{\mathcal{R}(\Omega) \mid \Omega \subset \mathbf{D}^n \text{ open}\}$, by assigning to every open subset Ω of $\mathbf{D}^n$ the n-th relative cohomology group with support in Ω and coefficients in $\widetilde{\widetilde{\mathcal{O}}}$:

$$\mathcal{R}(\Omega) \equiv H^n_\Omega(U, \widetilde{\widetilde{\mathcal{O}}}),$$

for any complex neighbourhood U of Ω (see [MO] for a concise introduction to relative cohomologies and their use in hyperfunction theory). Let us just note aside, that in one dimension this definition reduces to simply taking equivalence classes: $\mathcal{R}(\mathbf{D}^1) = \widetilde{\widetilde{\mathcal{O}}}(\mathbf{Q}^1 \setminus \mathbf{D}^1)/\widetilde{\widetilde{\mathcal{O}}}(\mathbf{Q}^1)$. One finds with these definitions:

THEOREM. *The presheaf $\{\mathcal{R}(\Omega)\}$ is a flabby sheaf. Further, for any compact set K in $\mathbf{D}^n$ we have $\mathcal{R}(K) = H^n_K(U, \widetilde{\widetilde{\mathcal{O}}}) \simeq \mathcal{P}'_{**}|_K$. In particular $\mathcal{R}(\mathbf{D}^n)$ is isomorphic to the dual space of $\mathcal{P}_{**}$.*

See, e.g., [15], theorems 3.5 and 3.6. The second part is the fourier hyperfunction equivalent of the famous Martineau-Harvey duality theorem, which can be found in [MO], theorem 6.5.1.

Euclidean Reconstruction using modified Fourier hyperfunctions. The general strategy of reconstruction used in [15] is the same as the original one of Osterwalder-Schrader in the tempered case. Naturally, the temperedness estimate (RE) is to be replaced by a weaker one in consistence with the slowly increasing property (SI) of the boundary value representation of $\mathcal{P}'_{**}$. Indeed after formulating axiomatic quantum field theory with (modified) Fourier hyperfunctions[2] in [14] and [15], Nagamachi-Mugibayashi carry out the analytic continuation as outlined in section 2, and find for the corresponding Schwinger functions an estimate of the form

$$|\mathbf{S}_n(\xi)| \leq C^{(0)}_{n,\varepsilon} \cdot e^{\varepsilon |\xi|} \quad \forall \varepsilon > 0,$$

[2]Actually, they use Fourier hyperfunctions of a mixed type, which are ordinary Fourier hyperfunctions in the spatial variables and modified only in the time variables, but this shall not concern us here.

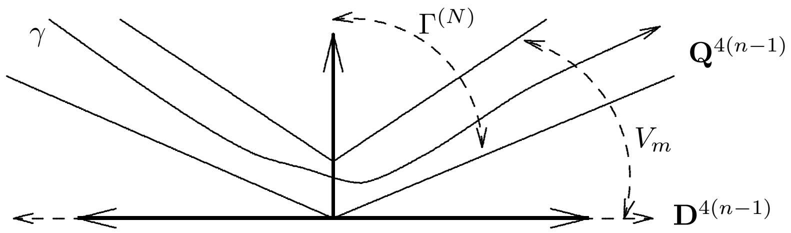
on the non-coincident Euclidean region, see theorems 4.2 and 4.6 in [15], which is there called *infra-exponential* estimate. Again, the real analytic functions $\mathbf{S}_n$ admit an analytic continuation to cones $\Gamma^{(N)}$ of increasing opening angle in the time variables and satisfiy a continued estimate analoguously to (NE):

$$|\mathbf{S}_n(\zeta)| \leq C_{n,\varepsilon}^{(N)} \cdot e^{\varepsilon|\zeta|},\ \forall\varepsilon > 0,\ \zeta \in \Gamma^{(N)}, \tag{NI}$$

see [15], proposition 5.3. Now, for the continued $\mathbf{S}_n$ to define a distribution on $\mathcal{P}_{**}(\mathbf{D}^{4(n-1)})$ as wanted, one has to show that the evaluation of $\mathbf{S}_n$ on any test function $f \in \mathcal{P}_{**}(\mathbf{D}^{4(n-1)})$ makes sense. By definition ($\mathcal{P}_{**}$) such a function will be in one of the spaces building the inductive limit, i.e., we can take $f_m \in \mathcal{O}_c^m(V_m)$, where V_m is a neighbourhood of $\mathbf{D}^{4(n-1)}$ of the same form as in (V_m), i.e., with finite opening angle above the real "axis". So, can we define the integral

$$\mathbf{S}_n(f_m) \equiv \int_{\gamma(N,m)} \mathbf{S}_n(\zeta) f_m(\zeta) d\zeta,$$

for a certain integration contour $\gamma = \gamma(N,m)$ near $\mathbf{D}^{4(n-1)}$, depending on the domains of holomorphy of both f_m and $\mathbf{S}_n$ (namely $\Gamma^{(N)}$)? Indeed such a contour exists as shown below:



It exists if the opening angle of $\Gamma^{(N)}$ around $\imath\mathbf{D}^{4(n-1)}$ is larger than that of the given V_m, that is, after *finitely many steps* $N(m)$ of analytic continuation. The integral $\mathbf{S}_n(f_m)$ is then clearly finite by the characterization of the test function ($\mathcal{O}_c^m$) and the estimate (NI), see theorem 3.15 in [15].

The point is, that this result can be reached after finitely many steps for every given test function. this makes it simply *unnecessary* to control the growth of the constant $C_{n,\varepsilon}^{(N)}$ in n and thereby in N. So what we have found is that Euclidean reconstruction of modified Fourier hyperfunctions is possible *without growth conditions.*

It may seem as if the *modification* of the Fourier hyperfunctions, namely the introduction of test functions of exponential decrease in domains of finite opening angle over $\mathbf{D}^n$ is just as *ad hoc* as the linear growth condition. But let me point out that in the sense of boundary values such a condition is dictated by the choice of compactification of $\mathbf{C}^n$, and the radial compactification, which gives equal importance to real and imaginary coordinates naturally leads to (V_m) and $(\mathcal{O}_c^m)$. We will compare this with ordinary Fourier hyperfunctions below.

4. Outlook

The two sorts of Euclidean reconstruction discussed above represent two extremes: The strong linear growth condition leading to the reconstruction of very “regular” — namely tempered — distributions, and the absence of growth conditions resulting in a very “singular” class of reconstructed functions — namely hyperfunctions — which can for example contain singularities like $exp(1/x)$. This phenomenon seems to be more of a fundamental than of a technical type. As is well known from the variants of the Payley-Wiener-Schwartz theorem one has as a rule of thumb that weaker growth conditions correspond to more singular distributions as boundary values[3], see, e.g., the discussion at he beginning of chapter 8 of [KA].

This naturally raises the question if one could find intermediate cases between these extremes. Florin Constantinescu has since long proposed to use *ultradistributions* in that connection, see [21], [22], [23], and the introductory text [24], but I want to point out another direction which could be taken.

Reconstruction of ordinary Fourier hyperfunctions. If one compactifies only $\mathbf{R}^n$ by adding the points at infinity to get $\mathbf{D}^n$ and uses $\mathbf{D}^n + i\mathbf{R}^n$ as the base space from which boundary values on the real axis are taken as hyperfunctions, one is led to the definition of ordinary Fourier hyperfunctions, also termed *Fourier hyperfunctions of type I* in [14], where the quantum field theory is formulated with the latter. On the side of the test function spaces the difference to the modified ones lies in the form of the complex neighbourhoods of the real axis in the equivalents of definitions (V_m) and $(\mathcal{O}_c^m)$. They have to be chosen as parallel stripes along the real axis with certain width in the imaginary direction:

$$U_{1;m} \equiv \{z \in \mathbf{C} \mid |\operatorname{Im} z| < 1/m\}.$$

One can the proceed to build the space $\mathcal{P}_*$ of test functions for Fourier hyperfunctions as an inductive limit as in $(\mathcal{P}_{**})$. One immediately sees $\mathcal{P}_{**} \subset \mathcal{P}_*$ since the elements of $\mathcal{P}_{**}$ have to satisfy growth conditions,and in fact to be analytic on larger domains as that of $\mathcal{P}_*$. By that, $\mathcal{P}'_{**}$ is a larger, i.e. “more singular,” class of distributions.

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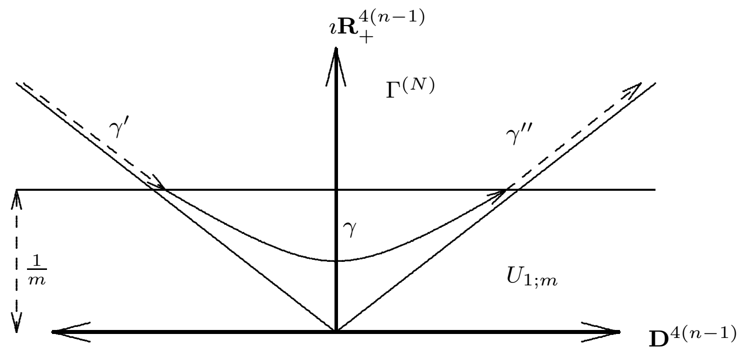
[3] Remember that in the Osterwalder-Schrader procedure the bound in n was used to get a bound in N, i.e., essentially in the distance from the real axis.

Trying to prove an Euclidean reconstruction theorem yielding boundary values in $\mathcal{P}'_*$ leads to the same difficulties as in the tempered case. Let us consider a fixed test function $f \in \mathcal{O}_c^m(V_{1;m}) \subset \mathcal{P}_*$. For this f, the integral $\mathbf{S}(f)$ with the analytic continuation of $\mathbf{S}_n$ can only be carried out partially on the finite integration path γ shown in the sketch below.



This shows that, as in the tempered case, $\mathbf{S}_n$ defines a functional on the test function space only after infinitely many steps of analytic continuation, calling again for growth conditions in N.

Maybe the situation is not so hopeless. One could try to approximate the test function f by certain functions f_k such that the integrals $\mathbf{S}_n(f_k)$ exist on the whole path $\gamma' \circ \gamma \circ \gamma''$. Such an approximation could possibly consist of quasi-analytic extensions of f out of the stripe $U_{1;m}$ as frequently used by Hörmander, see [HÖ]. For such extensions one has L^2-estimates, and one could hope to calculate what I would call the *functional error*

$$R(N, n; m, k) \equiv \left| \int_{\gamma'} + \int_{\gamma''} \mathbf{S}_n(\zeta) f_m(\zeta) d\zeta \right| .$$

which loosely speaking measures the distance from $\mathbf{S}_n$ to a functional on the whole space $\mathcal{P}_*$. Trying then to control $R(N, n; m, k)$ in m and k should make it possible by the same procedures as used by Osterwalder-Schrader to pose a condition on the original Schwinger functions like the linear growth condition that make $R(N, n; m, k)$ vanish in the limit $N \to \infty$ and thus is a sufficient condition for Euclidean reconstruction of Fourier hyperfunctions.

Intermediate spaces of hyperfunctions. One could think of constructing a sequence of spaces continuously mediating between $\mathcal{P}_*$ and $\mathcal{P}_{**}$ by simply modifying the geometry. Set for $0 \leq \rho \leq 1$

$$\begin{aligned} U_{\rho;m} &\equiv \{z \in \mathbf{C} | \, |\operatorname{Im} z| < (1 + |\operatorname{Re} z|)^\rho / m\}, \\ \mathcal{P}_\rho &\equiv \varinjlim_{m\to\infty} \mathcal{O}_c^m(V_{\rho;m}), \end{aligned} \qquad (\mathcal{P}_\rho)$$

where $V_{\rho;m}$ is the closure of $U^n_{\rho;m}$ in $\mathbf{Q}^n$ as in (V_m). The definition of the inductive limit can be easily shown to be independent of the exact form of $U_{\rho;m}$, i.e., wether one takes $(1+|\operatorname{Re} z|)^\rho/m$, $(1+|\operatorname{Re} z|^\rho)/m$ or $\{(1+|\operatorname{Re} z|)/m\}^\rho$ as the upper bound for the imaginary part of z in the definition. Then one finds the spaces of (modified) Fourier hyperfunctions as limit cases $\mathcal{P}_* = \mathcal{P}_0$ and $\mathcal{P}_{**} = \mathcal{P}_1$ in the sequence $\mathcal{P}_\rho \subset \mathcal{P}_{\rho'}$, $\rho < \rho'$, of *intermediate spaces.* The $\mathcal{P}_\rho$ can also be used as test function spaces for Fourier hyperfunctions:

LEMMA. *The Fourier transformation is a topological isomorphism on $\mathcal{P}_\rho$.*

This can be easily shown following the proof of proposition 3.2 in [13] with minor modifications.

As the intersection of $U_{\rho;m}$ with $\Gamma^{(N)}$ becomes larger as ρ increases the error term $R(N,n;m,k)$ should decrease, and it would be very tempting to get by the proceeding outlined above a continuous sequence of growth conditions which pose no restriction in the limit case $\mathcal{P}_1$ and tend to the — yet unknown — growth condition needed for reconstruction of distributions in $\mathcal{P}'_*$ in the limit $\rho \to 0$.

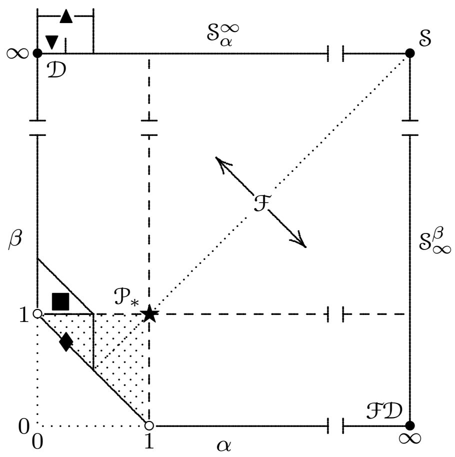
The scheme of Gelfand-Shilov spaces. In [GS], which is volume II of the famous series of books on functional analysis and distribution theory, Gelfand and Shilov introduced a general classification of test function spaces for distributions. They classified test functions by growth order in coordinates and derivatives by defining

$$\begin{aligned} \mathcal{S}_\alpha &\equiv \{f \in \mathcal{C}^\infty(\mathbf{R}) \mid |x^k f^{(q)}(x)| \le C_q A^k k^{k\alpha}\}, \\ \mathcal{S}^\beta &\equiv \{f \in \mathcal{C}^\infty(\mathbf{R}) \mid |x^k f^{(q)}(x)| \le C_k B^q q^{q\beta}\}, \\ \mathcal{S}^\beta_\alpha &\equiv \{f \in \mathcal{C}^\infty(\mathbf{R}) \mid |x^k f^{(q)}(x)| \le C A^k B^q k^{k\alpha} q^{q\beta}\}. \end{aligned} \qquad (\mathcal{S}^\beta_\alpha)$$

Within this scheme are contained many known examples of test function spaces such as $\mathcal{D}$ and $\mathcal{S}$, and they are set into relation by the well examined properties of the $\mathcal{S}^\beta_\alpha$ and their behaviour with respect to varying indices. The sketch below, whose details we will now briefly discuss, presents the $\mathcal{S}^\beta_\alpha$-scheme.

First of all, the spaces with only one index are limit cases of the general ones, i.e., we can extend the definition of $\mathcal{S}^\beta_\alpha$ to infinite values of α and β and find $\mathcal{S}^\beta = \mathcal{S}^\beta_\infty = \lim_{\alpha\to\infty} \mathcal{S}^\beta_\alpha$, $\mathcal{S}_\alpha = \mathcal{S}^\infty_\alpha = \lim_{\beta\to\infty} \mathcal{S}^\beta_\alpha$, and finally one recognizes $\mathcal{S}^\infty_\infty$ to be identical with the space of rapidly decreasing infinitely differentiable functions $\mathcal{S}$. One also finds the test function space $\mathcal{D}$ of $\mathcal{C}^\infty$-functions with compact support in the upper left corner, $\mathcal{D} = \mathcal{S}^\infty_0$, and its Fourier transform in the lower right: $\mathcal{F}\mathcal{D} = \mathcal{S}^0_\infty$.

This is, as one could already have guessed from $(\mathcal{S}^\beta_\alpha)$, a general phenomenon, showing the duality of the indices α and β with respect to Fourier transformation: $\mathcal{F}\mathcal{S}^\beta_\alpha = \mathcal{S}^\alpha_\beta$. As a special case we find that the spaces $\mathcal{S}^\alpha_\alpha$ on the diagonal are closed under Fourier transformation.



As the index β controls the derivatives of a function $f \in \mathcal{S}_\alpha^\beta$, it can also control the convergence of its Taylor series and by that the analytic continuation of f, if it exists. In fact, for $\beta \leq 1$, every such f can be analytically continued to some stripe around the real axis depending on B and characteristics of f. The stripes grow in width like $1/(eB)$ as B decreases and especially for $\beta < 1$, i.e., below the dashed horizontal line, $\mathcal{S}_\alpha^\beta$ consists of entire functions. This also means that the functions in $\mathcal{S}_\alpha^\beta$ with $\alpha < 1$ have Fourier transforms which are entire functions. See also [26] for a characterization of $\mathcal{S}_\alpha^\beta$ in terms of Fourier transforms.

It is apparent form the definitions that $\mathcal{S}_\alpha^\beta$ is contained in the intersection of $\mathcal{S}_\alpha$ and $\mathcal{S}^\beta$, but the other inclusion is also true as was shown in [27], so that we have $\mathcal{S}_\alpha^\beta = \mathcal{S}_\alpha \cap \mathcal{S}^\beta$. We also see that the Gelfand-Shilov spaces become smaller with decreasing indices, as sharper conditions are imposed on the functions. Indeed, $\mathcal{S}_\alpha^\beta$ can be shown to be trivial, i.e., contain only constants, for $\alpha+\beta < 1$, as well as are $\mathcal{S}_0^1$ and $\mathcal{S}_1^0$.

As already mentioned, models of interacting quantum fields can only be formulated in more singular classes of distributions than the tempered one, due to the singularities of the correlation functions which can be deduced from the constructive ingredients of the model, the Lagrangean, commutation relations, covariance, and so on. The suitable test function spaces for certain models were classified in the Gelfand-Shilov scheme in [25]:

- The *two-dimensional dipole field*, which is a solution of $\Box^2\varphi = 0$, has a two-point function which is a distribution on $\mathcal{S}_\alpha$, for $\alpha < 1/2$, marked by ▲.

- The *vertex operator* $:\exp(ig\varphi):$ of this field is even more singular, as it is an operator valued distribution over $\mathcal{S}_\alpha$, $\alpha < 1/4$, which is marked by ▼.
- $:\exp(ig\varphi):$ for the dipole field in 4 spacetime dimensions is — in the time variables — a distribution over $\mathcal{S}_\alpha^\beta$, $\alpha + \beta < 3/2$, $\alpha < 1/2$, the area marked by ■.

Most important for our discussion, also (modified) Fourier hyperfunctions can be arranged into the scheme. The test function space $\mathcal{P}_*$ of Fourier hyperfunctions was shown in [14] to be isomorphic to S_1^1, the point which is marked by ★. For $\mathcal{P}_{**}$, only a dense subspace could be found, see [15], lemma 2.1 and proposition 2.2., with $\mathcal{S}_{1/4}^{3/4}$ (or $\mathcal{S}_{3/4}^{1/4}$ since $\mathcal{P}_{**}$ is closed under Fourier transformation), which is marked by ⧫.

It should be possible to arrange the intermediate spaces $\mathcal{P}_\rho$ into the $\mathcal{S}_\alpha^\beta$-frame in a similar manner. This can give a first hint if growth conditions are needed for an Euclidean reconstruction on $\mathcal{P}'_\rho$, since the fact that $\mathcal{P}_{**}$ contains a dense subspace of *entire* functions seems to play a rôle in this respect. So far we have found:

Lemma. *For all α, $\beta \in (0,1)$ with $\alpha + \beta \geq 1$ and $0 \leq \rho \leq 1$ we have $\mathcal{S}_\alpha^\beta \subset \mathcal{P}_\rho$ for $\alpha \leq (1-\beta)/\rho$, and the topology of $\mathcal{S}_\alpha^\beta$ is stronger than that induced by $\mathcal{P}_\rho$ in this case. In particular $\mathcal{S}_\alpha^\alpha \subset \mathcal{P}_\rho$ for $\alpha \leq (1+\rho)^{-1}$.*

Proof. We use an alternative definition of $\mathcal{S}_\alpha^\beta$ as an inductive limit of locally convex spaces, which may be found in [15], section 2:

$$\begin{aligned} \mathcal{S}_\alpha^\beta &= \varinjlim_{n\to\infty} \mathcal{T}_\alpha^{\beta,n}, \\ T_\alpha^{\beta,n} &\equiv \{f \text{ entire} \mid ||f||_{(\alpha,\beta);n} < \infty\}, \quad \text{where} \\ ||f||_{(\alpha,\beta);n} &\equiv \sup_{z\in\mathbf{C}} |f(z)| e^{n^{-1}|\operatorname{Re} z|^{1/\alpha} + n|\operatorname{Im} z|^{1/(1-\beta)}}. \end{aligned}$$

Now it suffices to show that for every $f \in \mathcal{T}_\alpha^{\beta,n}$ we find some m such that $||f||_{\rho;m} \leq C||f||_{(\alpha,\beta);n}$, where $||.||_{\rho;m}$ is the norm introduced in $\mathcal{O}_c^m(V_{\rho;m})$ by formula ($\mathcal{O}_c^m$). Since then we have continuous embeddings $T_\alpha^{\beta,n} \hookrightarrow \mathcal{O}_c^m(V_{\rho;m})$ $\hookrightarrow \mathcal{P}_\rho$ and by that of $\mathcal{S}_\alpha^\beta$ in $\mathcal{P}_\rho$. To estimate the norms it is enough to estimate the exponents, i.e., to show

$$n^{-1}|\operatorname{Re} z|^{1/\alpha} + n|\operatorname{Im} z|^{1/(1-\beta)} \geq m^{-1}|z|, \quad \forall z \in U_{\rho;m},$$

for some m. It is enough to verify this condition on the boundary of the domain $U_{\rho;m}$, where the imaginary part becomes maximal, i.e., $|\operatorname{Im} z| = m^{-1}(1 +$

$\operatorname{Re} z)^\rho$. With that and using $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$, $a = \alpha^{-1}$, $b = (1-\beta)^{-1}$, $x \equiv |\operatorname{Re} z|$, we have only to show

$$n^{-1}x^a \geq nm^{-b}(1+x)^{\rho b} + m^{-1}x + m^{-1}(1+x)^\rho,$$

for $x > 0$ and a suitable choice of m. Since a, $b > 1$, $0 \leq \rho \leq 1$, this is always achieveable for sufficiently large m if $a \geq \rho b$, which is equivalent to $\alpha \leq (1-\beta)/\rho$, which completes the proof.

One would like to know if the largest of the subspaces of $\mathcal{P}_\rho$ lying on the diagonal, i.e., $\mathcal{S}^\alpha_\alpha$, $\alpha = (1+\rho)^{-1}$ is dense in $\mathcal{P}_\rho$, but this has yet to be examined.

Related Work. In [28] Yury M. Zinoviev established a full equivalence between a certain formulation of Euclidean and the Wightman field theory by inventing a new inversion formula for the Fourier-Laplace transformation for tempered distributions. But he had to impose an additional condition on the Euclidean side which he calls *weak spectral condition*, and whose physical meaning is somewhat unclear to me.

A further extension of distribution theory building on the Glefand-Shilov classification but exceeding it, was proposed as an extension of ultradistribution and hyperfunction theory suitable for the formulation of gauge quantum field theories in [29]. This extension corresponds to spaces $\mathcal{S}^\beta_\alpha$ with $\beta < 1$ in momentum space in accordance with the classification of some models, which was discussed above, in configuration space. He proves a sort for localizability of distributions on these spaces and other technical tools needed for quantum field theory. This extension is applied to the rigorous definitionof Wick-Ordered entire functions of free fields in [30].

To conclude let us say that the question of the 'right' test function spaces for interacting quantum fields is still unresolved, but at least *modified* hyperfunctions for which, as we have shown above $\mathcal{S}^{3/4}_{1/4}$ is a suitable test function space, are a reasonable one candidate. This is because these hyperfunctions allow to include all physical models discussed above, and equally important the beautiful theoretical machinery of hyperfunctions is at hand for them. On the other hand one should not be surprised if it needs further generalizations as in [29,30] if one wants to go over from 'toy–models' to 'real–world problems'.

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Andreas U. Schmidt
Fachbereich Mathematik
Johann Wolfgang Goethe–Universität
D–60054 Frankfurt am Main
e–mail:
aschmidt@math.uni–frankfurt.de